Circuit Complexity of Bipartite Matching in Grid Graph

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***Abstract*— We aim to show that Bipartite Matching for grid graphs is in AC^0. Recent works have shown that Bipartite Matching for grid layered planar graphs is in ACC^0. We aim to extend their work. We first describe the reduction of Bipartite Matching in grid layered planar graph to a word problem over monoids. Then we attempt to show the monoids obtained by reduction will always be aperiodic. Proving monoid obtained by reduction to be aperiodic reduces to a graph theoretic problem by extending the techniques used by Hansen et al. [2]. We will give proofs for special cases of this graph theoretic problem. Later we will look at other approaches and attempts for a general solution of this problem.**

I. Motivation

In a recent work, Hansen et al. [2] have shown that Bipartite matching for grid layered planar graphs is in ACC^0. They reduce the Bipartite Matching for Grid Layered Planar Graphs to word problem over monoid. For solvable monoids, word problem is in ACC^0 [1]. So, Hansen et al. [2] shows that reduced monoids will always be solvable. For aperiodic monoids the problem is in AC^0 [1]. We attempt to show that reduced monoids will always be aperiodic for grid graphs. This will put bipartite matching for grid graphs in AC^0. Proving reduced monoids are always aperiodic reduces to a graph theoretic problem. This technique for reduction was introduced in Hansen et al. [2] to show the monoids are solvable. This same method can be extended to formulate a sufficient condition for aperiodic monoids. We first look at the reduction given by Hansel et al.[2]

II. Definitions

A. Grid Graphs

Throughout this report, by grid graph, we mean a planar grid graph. We consider a grid $\wedge = \{1, \ldots, l\} \times \{1, \ldots, w\}$ of width w and length l. A grid graph $G = (V, E)$ of width w and length l is a graph where $V \subseteq \wedge$ and all edges are of Euclidean length 1[2].

B. Grid Layered Planar Graph

A Grid Layered Planar Graph $G = (V, E)$ of width w and length l is a graph embedded in the plane with no edge-crossings, with $V \subseteq \wedge$ and if two vertices (a, b) and (c, d) are connected by an edge, then $|a - c| \leq 1$[2].

C. Monoids

A monoid, $\mathcal{M}$, is a set with an associative binary operation and a two sided identity.

D. Word Problem over Monoids

Given $X_i \in \mathcal{M}$ for $i \in \{1, \ldots, n\}$, find product of all X_i's.

E. Solvable Monoids

A monoid, $\mathcal{M}$, which contains only solvable groups is called solvable. Word problem for such monoids is in ACC^0.[1]

F. Aperiodic Monoids

A monoid, $\mathcal{M}$, which contains only trivial groups is called solvable. Word problem for such monoids is in AC^0.[1]

III. Reduction to Monoid Word Problem

Hansen et al [2] showed that Bipartite Matching for Grid layered planar graphs can be reduced to Monoid Word problem. We present the monoid from their reduction. We define for a grid layered planar graph G with w rows,

Now we define monoid as,

$$\mathcal{M} = \{G^{\mathcal{M}} : \text{G is an odd length grid layered planar graph}\} \cup \{1\} \cup \{0$$

We define binary operation for monoid as follows:- $(X_1, Y_1, R)(X_2, Y_2, S) = (X_1, Y_2, R \circ S)$ for $Y_1 = X_2$.
Otherwise, $(X_1, Y_1, R)(X_2, Y_2, S) = 0$.
1 is identity and $x0 = 0x = 0$.
To show the Bipartite matching is in ACC^0 for grid layered planar graphs Hansel et al [2] showed that $\mathcal{M}$ is a solvable monoid. We follow there technique to obtain a sufficient condition for $\mathcal{M}$ to be aperiodic monoid.
Consider a group $\mathcal{G} \subset \mathcal{M}$. As $\mathcal{G}$ is a group, if $(X_1, Y_1, R) \in \mathcal{G}$ and $(X_2, Y_2, S) \in \mathcal{G}$ then $X_1 = X_2 = Y_1 = Y_2$ as $0 \notin \mathcal{G}$. So we can represent (X, Y, R) by R. Suppose $\mathcal{G}$ is not trivial then there exists a $R \in \mathcal{G}$ of order $o > 1$. Let E be identity in $\mathcal{G}$.

Theorem 1: If $E \subseteq R^{o-1}$, then $o = 1$.

Proof:

$$E \subseteq R^{o-1} \implies ER \subseteq R^o \implies R \subseteq E \implies R^{o-1} \subseteq E$$

Therefore $E = R^{o-1} \implies o = 1$. ■

Suppose R comes from a grid layered planar graph with w rows. Let $k = 2^w$ and $(Y_0, Y_{k+1}) \in E$ be any element of E. As $E^{k+1} = E$, there exists $Y_1, Y_2, \ldots, Y_k$ such that $(Y_i, Y_{i+1}) \in E$ for $i = 0, 1, \ldots, k$. Thus there exists $Y_i = X_1$ such that $(X_1, X_1) \in E$, $(Y_0, X_1) \in E$ and $(X_1, Y_{k+1}) \in E$. We will show that $(X_1, X_1) \in R^{o-1}$. As $R^{o-1} = ER^{o-1}E$, this gives $(Y_0, Y_{k+1}) \in R^{o-1}$ which implies $E \subseteq R^{o-1}$.

$(X_1, X_1) \in E$ and $R^o = E$. Thus there exists $X_2, X_3 \dots X_o$ such that $(X_i, X_{i+1}) \in R$ for all i and $(X_o, X_1) \in R$. Consider a graph G defining R Let M_i be matching on G for (X_i, X_{i+1}) for $i < o$ and M_o for (X_o, X_1). Let graph $S_i = M_i \cup M_{(i+1)}$ for $i < o$ and $S_o = M_o \cup M_1$. Let $S^n = S_1 \oplus S_2 \oplus \dots S_o \oplus S_1 \dots$ n times. We call each S_i a block. Let M be concatenation of matchings $M_1M_2 \dots M_oM_1M_2 \dots$ and N be concatenation of matchings $M_2M_3 \dots M_oM_1M_2 \dots$. We can clearly see that S^n is union of these two matchings restricted to first n block. Note that both M and N are valid matchings in first n blocks for S^n. Suppose $n = mo$ and S^n does not have a path from leftmost layer to rightmost layer. Let V_R be set of vertices reachable by rightmost layer. There is no edge from V_R to $\overline{V_R}$. Use matching edges from M for $\overline{V_R}$ and matching edges from N for V_R. Thus, we get $(X_1, X_2) \in R^n = E$ on this matching. This gives $(X_1, X_1) \in R^{o-1} = ER^{o-1}$ as $(X_1, X_2) \in E$ and $(X_2, X_3), (X_3, X_4), \dots, (X_o, X_1) \in R$. Thus, $R^{o-1} \subseteq E$. Therefore, we need to prove that for some $n = mo$, S^n does not have a path from leftmost layer to rightmost layer. We aim to show a slightly stronger result that for all $n > n_0$ and matching $M = M_1M_2 \dots$ and $N = M_2M_3 \dots$, S^n does not have a path from rightmost layer to leftmost layer.

IV. Notations

Consider an infinite grid graph, G, with n number rows and infinitely many columns. We divide the graph G into G_1, G_2, G_3,... such that G_i have n rows and l columns (l is odd) and

$$G = G_1 \oplus G_2 \oplus \dots$$

where $\oplus$ is graph concatenation operator. Let M_i be a perfect matching on G_i for $i > 1$ and for M_1 be a perfect matching on G_1 for all vertex except first vertices of first column. We call edges of M_i in G_i matching edges. Also project back edges in M_i to G_{i-1} for $i > 1$. We call these edges as projected edges. Let $\widetilde{G_i}$ be graph G_i with matching and projected edges. Also

$$\widetilde{G} = \widetilde{G_1} \oplus \widetilde{G_2} \oplus \dots$$

. We need to show that $\widetilde{G}$ cannot have infinite length path for all n, for all odd l, and, for all M_i's. Proving this will put the Bipartite Matching for grid graph in AC^0.

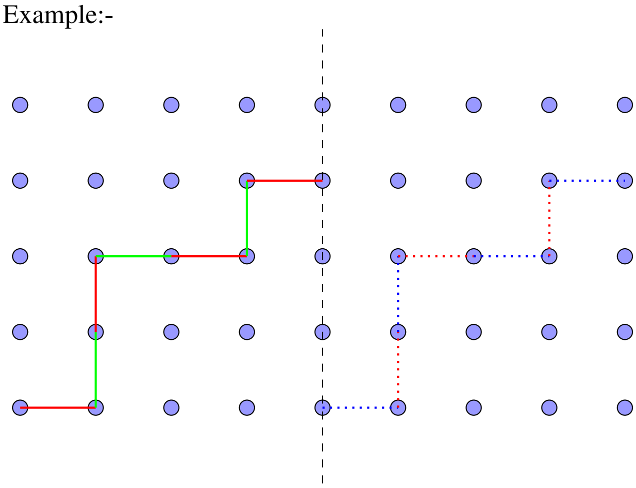
V. Important Properties and Definitions

A. Degree of vertices in $\widetilde{G}$

Any vertex in $\widetilde{G}$ can have atmost two edges(one matching edge and one projected edge). Thus degree of any vertex is atmost 2. Therefore we can have closed loops, path segments and infinite paths only. Also no two matching edge be adjacent. Thus only projected edge can be adjacent to matching edge and vice versa.

B. Projected Path

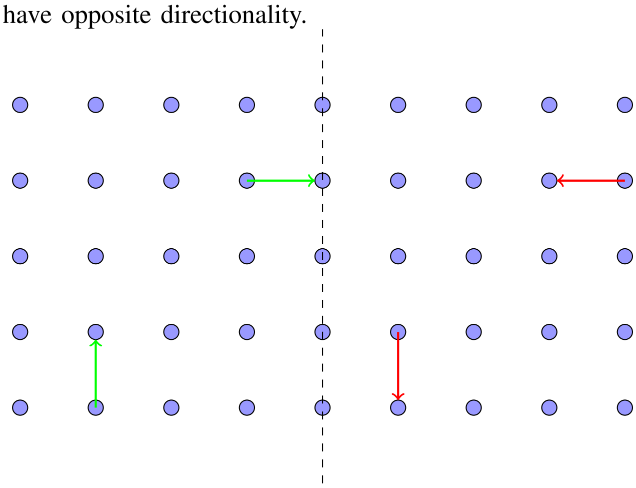
For any path $P = v_0, v_1, \dots, v_k$, we define its corresponding Projected path, $\widehat{P}$ as follows. For each edge (v_i, v_{i+1}) if it is a projected edge we add its corresponding matching edge (w_i, w_{i+1}) in $\widehat{P}$ and if it is a matching edge in $\widetilde{G}_j$, we *forward* project it to $\widetilde{G}_{j+1}$, and add this new *imaginary* edge to $\widehat{P}$.
Example:-



Solid edges form path, P and dotted edges forms corresponding projected path, $\widehat{P}$. Red and green colored edges show matching edges and projection edges in path P respectively. Red and blue colored edges show matching edges and forward projected edges in projected path, $\widehat{P}$, respectively.

C. Directionality of Matching Edges and Corresponding projected Edge

A matching edge and its corresponding projected edge have opposite directionality.



D. Monotonicity of Path

We can view grid graph as a cartesian plane labeling each vertex as (x, y). For a vertex v, we represent it's x-coordinate by x(v) and y-coordinate by y(v). We say a path, $P = v_0, v_1, \ldots v_k$, is horizontally monotonic if $x(v_0), x(v_1), \ldots x(v_k)$ is monotonically(not strictly) increasing or decreasing sequence. For simplicity we will call a horizontally monotone path as H-monotone. We say path, $P = v_0, v_1, \ldots v_k$, is piecewise vertically monotone if path continuously goes down to bottom row and then continuously up to top row and so on. For simplicity we will call such a path as V-monotone.

E. Closed Regions

A region is defined as set of vertices. A set of vertices which have to be matched among themselves is called closed region. Closed regions might be very helpful in solving the problem. For ex. existence of closed region with odd number of points will imply non-existence of proper matching for that region and thus for G. We will use this in proofs for special cases.

VI. Inductive Solution Approach

A. Proof for $n = 2$ and 3

Our initial approach to solution involved trying induction on n or l. We identified that a simple proof exists for $n = 2$. Taking that as base case we tried to give an inductive argument for larger n. This approach however was unsuccessful.

We will show non-existence of vertical edges in infinite path. This will prove the claim. Suppose we have a vertical matching edge in infinite path then we will have two projected horizontal edges just before and after the vertical matching edge. Let us look at matching edge corresponding to these edges. We can clearly see that by directionality these matching edges cannot let infinite path to cross them. Thus we cannot have a vertical matching edge. If we have a vertical projected edges, then we must have its corresponding vertical matching edge in infinite path as well. Thus we cannot have a vertical projected edge. This proves that we cannot have a vertical edge in infinite path. This shows that infinite path cannot exists for $n = 2$. We proved for $n = 3$ by a detailed case-by-case analysis. We also used computer simulation to further verify for $n = 3$.

Extending the proof similarly for $n = 4$ seemed non-trivial.

VII. Results for Monotonic paths

A. Proof for non-existence of H-monotone and V-monotone infinite paths in $\widetilde{G}$

Once inductive approaches seemed non-trivial, we started to put restriction on infinite paths. First we showed the non-existence of H-monotone and V-monotone infinite paths in $\widetilde{G}$. We then started to reduce monotonicity constraints.

Let us consider a segment S of an H and V monotone path going from the top row to the bottom row. This segment leads to the formation of a closed region C. The left boundary of C is formed by the matching edges of S and the right boundary of C is formed by the matching edges which generate the projected edges of S. We show that the number of vertices in C is odd.

The segment S enters and exists the i^{th} row exactly once by means of vertical edges for i ranging from 2 to $n - 1$. Let $C(i)$ denote the number of vertices in C present in the ith row. Based on the nature of these vertical edges, we can form 4 cases.

Case 1: Enters by a matching edge and exits by a matching edge - $C(i) = 0 \mod 2$

Case 2: Enters by a projected edge and exits by a projected edge - $C(i) = 0 \mod 2$

Case 3: Enters by a projected edge and exits by a matching edge - $C(i) = 1 \mod 2$

Case 4: Enters by a matching edge and exits by a projected edge - $C(i) = 1 \mod 2$

Now let us consider the first row. It can be shown that if S leaves the first row with a matching edge, then $C(1)$ is odd, else $C(1)$ is even.

Combining the results of the 4 cases with the above, we get

$$\sum_{i=1}^{n-1} C(i) = \begin{cases} 0 \mod 2, & \text{S enters the last row with} \\ & \text{a matching edge} \\ 1 \mod 2, & \text{otherwise} \end{cases}$$

However, the last row experiences a parity flip. It can be seen from the figure that $C(n)$ is odd if S enters last row with matching edge and even if it enters via a projected edge. This happens because last point in segment S is unmatched by path unlike other points on segment

Hence for all the cases $\sum_{i=1}^{n} C(i) = 1 \mod 2$.

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B. Removing horizontal monotonicity of path from previous proof

We present a stronger result by eliminating the need for H-monotone path.

The proof idea is essentially same as the previous case. The closed region C and the segment S are defined exactly same as above. Removing H-monotonicity still ensures that the segment S will enter and exit the ith row exactly once by means of vertical edges. In between these vertical edges, S can go strictly right or strictly left. The case by case analysis and the derivation from A can be extended to only V-monotone paths too.

C. A more stronger result

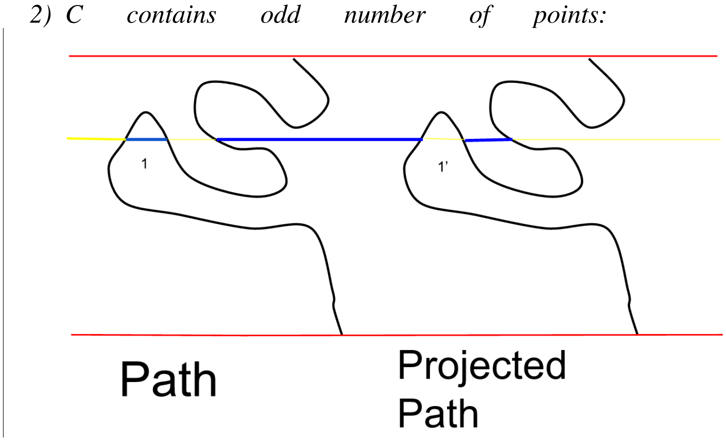
We removed the need for V-monotonicity as well by replacing with a weaker constraint. We present proof for non-existence of infinite path which goes from top to bottom row in $\widetilde{G}$.

We define closed regions as follows. For i^{th} row we define $C(i)$ to points in closed regions. Suppose we draw a ray from first column to infinity in i^{th} row. If ray falls at a point,p, after hitting odd number of vertical edges then $p \in C(i)$

otherwise $p \notin C(i)$. Union of $C(i)$ for all i makes closed region C.

1) C is a closed region: To show that C is a closed region note that any path P, corresponding projected path $\widehat{P}$, top row and bottom row together make polygon(s). A point x lies in polygon iff horizontal line passing through x hits edges of polygon odd number of times. Thus points in C are enclosed by matching edges, top row and bottom row. Therefore C is a closed region.

2) C contains odd number of points:



Notice that 1 and $1'$ are identical. Thus if we count $1'$ we will get same result. Thus counting $C(i)$ is same as counting number of points from rightmost vertical edge of path P in i^th row to corresponding edge in Projected Path $\widehat{P}$. Thus for all i, $C(i)$ contains even number of points except last for last row. Last row once again suffers a parity change and end point of path P in bottom row will be unmatched if we enter with a projected edge. If we enter with matching edge note that region is not closed so we extend path P to include a horizontal projected edge which then makes region closed but introduce and unmatched point in last row. Thus C must contain odd number of points. This concludes the proof.

VIII. Future Works

We need to generalize our proof our any grid graph as our proof works for complete grid graphs only. A weaker problem to look at might be to show that bipartite matching is in $AC^0[m]$ for some m. For this we need to show that every group $\mathcal{G} \subset \mathcal{M}$, have order m^k for some k. A more obvious directions is to extend proof for any path in complete grid graphs. For this we may look at other closed regions. This also raises a problem of what other type of closed regions(other than odd number of points) do not have perfect matching. If this keeps on failing, we should try to show that problem is AC^0-Hard.

IX. Conclusion

Although problem seemed quite difficult(failure of inductive proofs), we have have managed to solve some specific cases and we need some meaningful extension of results. Our works have increased understanding of problem and raised important questions like what structure of closed regions prevent matching.

References

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